# Bi-rotary Maps of Negative Prime Power Euler Characteristic

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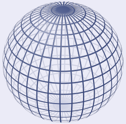
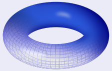
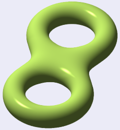
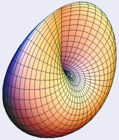
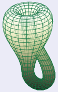
# Graphs and Surfaces

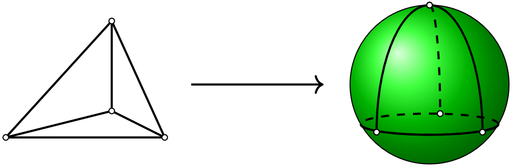
- Graph: an ordered pair $(V, E)$. Loops, parallel edges are allowed.
- (Closed) Surfaces: compact, connected and closed $2$-dimensional manifold.

**Theorem (Classification of closed surfaces)**

| *genus g* | *0* | *1* | *2* | . . . |
|---|---|---|---|---|
| *Orientable* |  |  |  | ... |
| *Non-orientable* | *None* |  |  | . . . |

# What is a map?

A map $\mathcal{M}$ is an embedding of a graph $\Gamma = (V, E)$ on a surface $\mathcal{S}$ such that each connected component of $\mathcal{S} \setminus (V \cup E)$ is homeomorphic to an open disk. $\Gamma$ is called the underlying graph of $\mathcal{M}$.



$\Gamma = \mathbf{K}_4$ $\qquad$ $S^2$

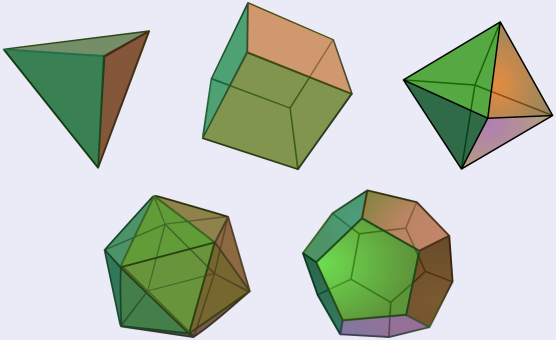
# What is a map?

- Vertex set $V$ of $\mathcal{M}$: the set of vertices of the underlying graph $\Gamma$.
- Edge set $E$ of $\mathcal{M}$: the set of edges of the underlying graph $\Gamma$.
- Face set $F$ of $\mathcal{M}$: the set of connected components of $\mathcal{S} \setminus (V \cup E)$.
- Euler characteristic of $\mathcal{M}$: $\chi = |V| - |E| + |F|$.

# What is a map?

A Platonic solid can be seen as a map on sphere ($\chi = 2$).

## Five Platonic solids



# What is a map?

### The Euler-Poincaré Formula

The Euler characteristic $\chi$ of a map $\mathcal{M}$ is determined by the surface $\mathcal{S}$ on which $\mathcal{M}$ is embedded:

$$\chi = |V| - |E| + |F| = \begin{cases} 2 - 2g & \text{if } \mathcal{S} \text{ is orientable,} \\ 2 - g & \text{if } \mathcal{S} \text{ is non-orientable} \end{cases}$$

where $g$ is the genus of $\mathcal{S}$.

# Combinatorial maps and symmetry

- Combinatorial map: Given a map $\mathcal{M}$ with vertex set $V$, edge set $E$ and face set $F$, the triple $(V, E, F)$ with the incidence relations is called a combinatorial map.
- Symmetry (Automorphism) of a map: a bijection on $V \cup E \cup F$ preserving $V, E, F$ and the incidence relations.
- $\mathrm{Aut}(\mathcal{M})$: the group consists of all automorphisms of the map $\mathcal{M}$.

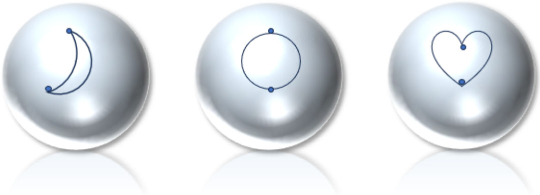


Figure: Maps that embedd digon into $S^2$

They have same combinatorial map and symmetries $\mathrm{Aut}(\mathcal{M}) = \mathbb{Z}_2^3$.

# Highly symmetric maps

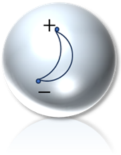
- Flags: incident triples of vertex-edge-face in a map.
- Regular map $\mathcal{M}$: $\mathrm{Aut}(\mathcal{M})$ acts transitively on flags.

Regular maps on

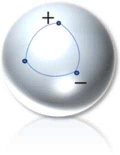
1. Sphere: the five Platonic solids, embedded $\ell$-cycles and its dual.
2. Projective plane: embedded $\ell$-cycles, $K_4$, $K_6$ and their duals.
3. Torus: quotients of Euclidean tessellations.
4. Klein bottle: None

# Highly symmetric maps

- Arcs: incident pairs of vertex-edge in a map.
- Bi-orientation: an assignment of local orientations at vertices such that adjacent vertices have opposite orientations.
- $\mathrm{Aut}^b(\mathcal{M})$: the group consist of all bi-orientation-preserving automorphisms of the bi-orientable map $\mathcal{M}$.
- Bi-rotary map $\mathcal{M}$: $\mathrm{Aut}^b(\mathcal{M})$ acts transitively on arcs.




Bi-orientable map




Non-bi-orientable map

# Highly symmetric maps

- Edge-transitive map $\mathcal{M}$: $\mathrm{Aut}(\mathcal{M})$ acts transitively on edges.
- (Graver & Watkins, 1997). There are exactly 14 classes of edge-transitive maps according to the action of $\mathrm{Aut}(\mathcal{M})$ on flags.
- Regular maps and bi-rotary maps are two of the 14 classes of edge-transitive maps.

# Breakthrough

- In 2005, Breda, Nedela and Širáň classified regular maps on all surfaces of negative prime Euler characteristic.
- In 2010, Conder, Potočnik and Širáň gave a classification of regular maps on surfaces of Euler characteristic $-p^2$.
- In 2010, Conder, Širáň and Tucker gave a complete classification of orientably regular maps on surfaces of Euler characteristic $-2p$.
- In 2019, Breda, Catalano and Širáň gave a classification of bi-rotary maps on surfaces of Euler characteristic $-p$.

# The classification problem of bi-rotary maps

**Question**

Can we classify bi-rotary maps on surfaces of Euler characteristic $-p^n$ for a prime $p$ and a positive integer $n$?

**Theorem (Breda et. al., 2019)**

Classification of bi-rotary maps on surfaces of Euler characteristic $\chi$ is equivalent to classification of finite groups $X = \langle x, y \rangle$ with $|y| = 2$ and

$$|X| \left( \frac{1}{|x|} + \frac{1}{2|[x, y]|} - \frac{1}{2} \right) = \chi.$$

Given such a group $X = \langle x, y \rangle$, denote the corresponding bi-rotary map by $\mathrm{Map}(X, x, y)$.

## Example (Macbeath's trick)

Let $D = \langle x, y | x^3, y^2, [x, y]^4 \rangle$ be a finitely presented group. This gives an infinite bi-rotary map $\mathcal{U} = \mathrm{Map}(D, x, y)$. There is a normal subgroup $N$ of $D$ such that $D/N = T \cong \mathrm{PSL}(2, 7)$ and this gives the quotient map $\mathcal{U}/N$, which is a bi-rotary map of Euler characteristic $-7$. It follows that

$$N \cong \pi_1(\mathcal{N}_9) = \langle a_i (1 \leq i \leq 9) \mid a_1^2 a_2^2 \ldots a_9^2 \rangle.$$

For any integer $n$, set

$$M_n = N' \mho_{7^n}(N), \text{ where } \mho_{7^n}(N) = \langle g^{7^n} | g \in N \rangle.$$

Then $N/M_n \cong \mathbb{Z}_{7^n}^8$ and $\mathcal{U}/M_n$ is a bi-rotary map with automorphism group $\mathbb{Z}_{7^n}^8.\mathrm{PSL}(2, 7)$ and Euler characteristic $-7^{8n+1}$.

# Methods

Let $X$ be a finite group and let $p$ be a prime.

1. $\mathscr{P}_0(p)$-group: $X = \langle x, y\rangle$ with $|y| = 2$ and
$$|X|\left(\frac{1}{|x|} + \frac{1}{2|[x,y]|} - \frac{1}{2}\right) = -p^n.$$
2. $\mathscr{P}_1(p)$-group: $X$ with two subgroups $H_1$ and $H_2$ which are cyclic or dihedral such that
$$|X| = p^n \cdot \mathrm{lcm}(|H_1|, |H_2|).$$
3. $\mathscr{P}_2(p)$-group: $X$ is a group such that for each prime $r \neq p$, the Sylow $r$-subgroup of $G$ is cyclic or dihedral.

**Lemma**

- $\mathscr{P}_0(p) \Rightarrow \mathscr{P}_1(p) \Rightarrow \mathscr{P}_2(p)$.
- $\mathscr{P}_1(p)$ is closed under normal subgroups and quotients.
- $\mathscr{P}_2(p)$ is closed under subgroups and quotients.

# Methods

**Idea:** Describe $\mathscr{P}_2(p)$-groups $\to$ Describe $\mathscr{P}_1(p)$-groups $\to$ Describe $\mathscr{P}_0(p)$-groups.

**Theorem (Our work)**

If $X$ is a $\mathscr{P}_0(p)$-group, then $G = X/O_p(X)$ can be classified. Moreover, if $X$ is abelian, then $X$ can be explicitly described.

$O_p(X)$: the largest normal $p$-subgroup of $X$.

# Main results: abelian case

**Theorem**

*Let $\mathcal{M}$ be a bi-rotary map with $n$ edges and an abelian automorphism group. Then either $\mathcal{M} \cong \mathcal{B}_n$ or $\mathcal{M} \cong \mathcal{D}_n$.*

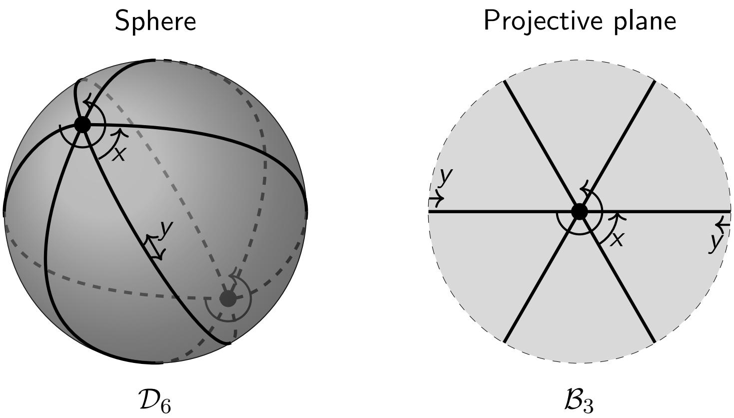



Figure: Two bi-rotary maps with abelian automorphism groups

# Main results: non-abelian and solvable case

**Theorem**

*If $G = X/O_p(X)$ is non-abelian and solvable, then:*

$$G = \langle a \rangle : (\langle b \rangle \times H)$$

*where $H$ is a Hall $\{2,3\}$-subgroup. All possible $H$ and types are listed in the table.*

| $H$ | $(\bar{k}, \bar{m})$ | $(\rho, \tau)$ | Comments |
|---|---|---|---|
| $Z_{k_1} = \langle \rho_0 \rangle$ | $(k_1 k_2 m_2', 2m_2)$ | $(ab\rho_0, \rho_0^{k_1/2})$ | |
| $Z_{k_1} \times Z_2 = \langle \rho_0 \rangle \times \langle \tau_0 \rangle$ | $(k_1 k_2 m_2', 2m_2)$ | $(ab\rho_0, \tau_0)$ | $p = 2$ |
| $D_{2\cdot 3^e} = \langle c \rangle : \langle d \rangle$ | $(3^e k_2 m_2', 2\cdot 3^e m_2)$ | $(abc, d)$ | $p = 2$ |
| | $(2k_2, 2\cdot 3^e m_2)$ | $(abcd, d)$ | $p \neq 3$ |
| $Z_{2^f} \times D_{2\cdot 3^e} = \langle d_1 \rangle \times \langle c \rangle : \langle d_2 \rangle$ | $(2^f 3^e k_2 m_2', 2\cdot 3^e m_2)$ | $(abcd_1, d_2)$ | $p = 2$ |
| | $(2^f k_2 m_2', 2\cdot 3^e m_2)$ | $(abcd_1 d_2, d_2)$ | |
| $Z_2^2 : Z_{3^e} = (\langle d_1 \rangle \times \langle d_2 \rangle) : \langle c \rangle$ | $(3^e k_2, 4)$ | $(bc, d_1)$ | $p \neq 2$ and $a = 1$ |

TABLE 1. Possible structures for $H$ and standard generating pairs for $G$

# Main results: non-solvable case

## Theorem

*If $G = X/O_p(X)$ is non-solvable, then:*

$$G = (R \times D).\mathbb{Z}_f$$

*where $f \leq 2$, $D \cong \mathrm{PSL}(2,q)$, and:*

(i) $p > 2$, $f = 1$, *R cyclic of odd order*, $q = 2p^t \pm 1$ *or* $q = p^t$

(ii) $p = 2$, $f = 1$, $q \in \mathcal{N}$ *(special numbers)*

(iii) $p = 2$, $f = 2$, $G = R \times (\mathrm{PSL}(2,q).\mathbb{Z}_2)$, $q \in \mathcal{N}$

(iv) $p = 2$, $f = 2$, $O_2(G/D) = 1$, $q \in \mathcal{N}$

Where $\mathcal{N}$ contains powers of 2 ($\geq 4$), Mersenne primes, and Fermat primes.