Embedding a Praeger-Xu graph into a surface

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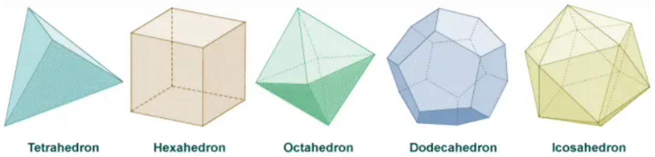
A map is a 2-cell embedding of a graph $\Gamma = (V, E)$ into a closed surface $\mathcal{S}$.



Underlying graph: Γ.
Supporting surface: $\mathcal{S}$.
Vertex set: V
Edge set: E
Face set: Components of $\mathcal{S} \setminus \Gamma$. Often denoted by F.

If no face is incident more than once with any edge or any vertex, then the map can be described combinatorially by (V, E, F).

Let $\mathcal{M} = (V, E, F)$ be a map, $\mathcal{S}$ be the supporting surface.

Aut($\mathcal{M}$): set of permutations of $V \cup E \cup F$ that preserve incidences.
Orientable maps: $\mathcal{S}$ is orientable.
Aut$^+$($\mathcal{M}$): set of automorphisms that preserve the orientation of $\mathcal{S}$.
Rotary maps: $\mathcal{M}$ is orientable and Aut$^+$($\mathcal{M}$) acts regularly on arcs (darts) of Γ.

Praeger-Xu graphs (PX graphs)

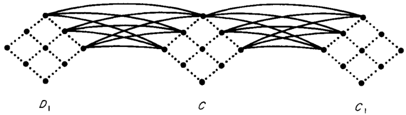
Let p, r, s be positive integers such that $p \geqslant 2$ and $r \geqslant 3$. Define a simple graph $\mathrm{C}(p, r, s) = (V, E)$ as follows:

1. the vertex set V is $\mathbb{Z}_r \times \mathbb{Z}_p^s$;
2. the edge set E is defined to be the set of all pairs of the form

$$\{(i, x_0, x_1, \ldots, x_{s-1}), (i+1, x_1, \ldots, x_{s-1}, x_s)\}$$

for every $i \in \mathbb{Z}_r$ and $x_0, x_1, \ldots, x_{s-1}, x_s \in \mathbb{Z}_p$.

$\mathrm{C}(p, r, s)$ has $p^s r$ vertices, $p^{s+1} r$ edges and valency $2p$.



A PX graph $\mathrm{C}(p, r, s)$ is an elementary abelian multicover of cycle C_r.

Theorem (D., Z. Guo and L. Liu, 2025+)

Let p odd prime, $r \geqslant 3$ such that $p \nmid r$. $\exists$ one-one correspondence between rotary maps with underlying graph $\mathrm{C}(p, r, s)$ and multiplicity-free representations of D_{2r} over $\mathbb{F}_p$ of degree $s + 1$ (except a few cases).

We always assume p is odd, $r \geqslant 3$ and $p \nmid r$ for the remainder.

Example

Let $p = 13$, $r = 7$ and $s = 3$. D_{14} has 5 irreducible representations $\psi_1, \ldots, \psi_5$ over $\mathbb{F}_{13}$, which is of degree $1, 1, 2, 2, 2$ respectively. Multiplicity-free representations of degree 4 are $\psi_1 + \psi_2 + \psi_3$, $\psi_1 + \psi_2 + \psi_4$, $\psi_1 + \psi_2 + \psi_5$, $\psi_3 + \psi_4$, $\psi_3 + \psi_5$ and $\psi_4 + \psi_5$. Thus there are 6 non-isomorphic rotary maps with underlying graph $\mathrm{C}(13, 7, 3)$.

(1) Find arc-regular groups of automorphisms G of Praeger-Xu graphs $\mathrm{C}(p, r, s)$. $(\mathrm{Aut}^{+}(\mathcal{M}) \leqslant \mathrm{Aut}(\mathcal{M}) \leqslant \mathrm{Aut}(\Gamma).)$

(2) Construct rotary maps with underlying graph $\mathrm{C}(p, r, s)$ through G.

(3) Classification of all rotary maps with underlying graph $\mathrm{C}(p, r, s)$.

(4) Construct the connection between rotary maps and representations of D_{2r} over $\mathbb{F}_p$.

Theorem ([4, Theorem 2.10])

$\mathrm{C}(p, r, s)$ is arc-transitive if and only if $r \geqslant s+1$, and is vertex transitive if and only if $r \geqslant s$.

We assume $r \geqslant s+1$ for the remainder.

Theorem ([4, Theorem 2.13])

$A := \mathrm{Aut}(\mathrm{C}(p, r, s)) = \mathrm{S}_p \wr_{[r]} \mathrm{D}_{2r}$ with an **arc-stabilizer** $A_{\alpha\beta}$ equal to $\mathrm{S}_p^{r-s-1} \times \mathrm{S}_{p-1}^{s+1}$ if $(r, s) \neq (4, 1)$ and $\mathrm{C}(p, r, s) = \mathrm{K}_{2p,2p}$ if $(r, s) = (4, 1)$.

If $G \leqslant A$ is arc-regular, then $A = GA_{\alpha\beta}$, $G \cap A_{\alpha\beta} = 1$ and $|G| = 2p^{s+1}r$.

[0] [4] C. E. Praeger and M. Y. Xu. A Characterization of a Class of Symmetric Graphs of Twice Prime Valency. 1989

Proposition (D., Z. Guo and L. Liu, 2025+)

If G is an arc-regular subgroup of $\mathrm{Aut}(\mathrm{C}(p, r, s))$, then

$$G \cong \mathbb{Z}_p^{s+1} \rtimes \mathrm{D}_{2r}.$$

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For convenience, we say it is a PX map if a rotary map has underlying PX graph.

Corollary

If $\mathcal{M}$ is a PX map, then $\mathrm{Aut}^+(\mathcal{M}) \cong \mathbb{Z}_p^{s+1} \rtimes \mathrm{D}_{2r}$.

Rotary pair: $(\rho, \tau) \in G \times G$ satisfies $G = \langle \rho, \tau \rangle$ and $|\tau| = 2$.
coset rotary map: RotaMap(G, ρ, τ) such that

$$V = [G : \langle \rho \rangle], E = [G : \langle \tau \rangle], F = [G : \langle \rho\tau \rangle]$$

where two objects are incident if and only if their set intersection is non-empty. We say that $\mathcal{M}$ is G-rotary if $G \cong \mathrm{Aut}^+(\mathcal{M})$.

Theorem ([3, Proposition 5.1])

RotaMap(G, ρ, τ) is a G-rotary map and a G-rotary map is isomorphic to RotaMap(G, ρ, τ) for some rotary pair (ρ, τ).

Theorem ([3, Proposition 4.1])

Two maps RotaMap(G, ρ, τ) and RotaMap(H, ρ', τ') are isomorphic if there is a group isomorphism $f\colon G \to H$ such that $f(\rho) = \rho'$ and $f(\tau) = \tau'$.

[0] [3] C. H. Li, C. E. Praeger, and S. J. Song. Locally finite vertex-rotary maps and coset graphs with finite valency and finite edge multiplicity. 2024.

Lemma (D., Z. Guo and L. Liu, 2025+)

Let $G = \mathbb{Z}_p^{s+1} \rtimes \mathrm{D}_{2r}$ with $s \geqslant 1$, and let $\mathcal{M} = \mathrm{RotaMap}(G, \rho, \tau)$. If $|\rho| = 2p$, then the underlying graph Γ is isomorphic to $\mathrm{C}(p, r, s)$. Moreover, all PX maps can be obtained in this way.

Lemma (D., Z. Guo and L. Liu, 2025+)

Let $G = \mathbb{Z}_p^{s+1} \rtimes \mathrm{D}_{2r}$ with $s \geqslant 1$, and let $\mathcal{M} = \mathrm{RotaMap}(G, \rho, \tau)$. If $|\rho| = 2p$, then the underlying graph Γ is isomorphic to $\mathrm{C}(p, r, s)$. Moreover, all PX maps can be obtained in this way.

Question: How to find all 'non-isomorphic' rotary pairs (ρ, τ) such that $|\rho| = 2p$?

Let $G = \mathbb{Z}_p^{s+1} \rtimes_\psi \mathrm{D}_{2r} = V \rtimes_\psi \langle x, y\rangle$ with $s \geqslant 1$, $|x| = |y| = 2$.

Irreducible case: ψ is irreducible. We call $\mathcal{M}$ is irreducible. In this case, for every $v \in \mathrm{C}_V(x) \setminus \{1\}$, (vx, y) is a rotary pair of G such that $|vx| = 2p$.
Let $\psi^{\mathrm{Aut}(\mathrm{D}_{2r})} := \{\psi \circ \sigma \mid \sigma \in \mathrm{Aut}(\mathrm{D}_{2r})\}$.

Proposition (D., Z. Guo and L. Liu, 2025+)

There is a one-one correspondence between $\psi^{\mathrm{Aut}(\mathrm{D}_{2r})}$ and non-isomorphic G-rotary PX maps.

Idea of proof

- There is an injective map f from $\psi^{\mathrm{Aut}(\mathrm{D}_{2r})}$ to non-isomorphic rotary pairs given by $\psi \circ \sigma \mapsto (v_\sigma \sigma(x), \sigma(y))$.
- Compute the number of rotary pairs (ρ, τ) such that $|\rho| = 2p$ and the size of $\mathrm{Aut}(G)$.
- There are exactly $|\psi^{\mathrm{Aut}(\mathrm{D}_{2r})}|$ non-isomorphic rotary pairs (ρ, τ) such that $|\rho| = 2p$. Hence f is bijective.

Note that $G \cong \mathbb{Z}_p^{s+1} \rtimes_{\psi_1} \mathsf{D}_{2r}$ iff $\psi_1 \cong \psi \circ \sigma$ for some $\sigma \in \mathrm{Aut}(\mathsf{D}_{2r})$.

Corollary

There is a one-one correspondence between $\mathsf{Irr}(\mathrm{D}_{2r})$ over $\mathbb{F}_p$ and non-isomorphic irreducible PX maps.

Reducible case: ψ is reducible. How to find all non-isomorphic rotary pairs (ρ, τ) such that $|\rho| = 2p$?

Idea: find 'nice' normal subgroups M of G such that $\rho, \tau \notin M$ and $G/M \cong \mathbb{Z}_p^{t+1} \rtimes \mathrm{D}_{2r}$ is irreducible.

By Maschke's theorem, $G = (V_1 \times \cdots \times V_n) \rtimes \mathrm{D}_{2r}$ where $V_i \lhd \mathrm{D}_{2r}$. Let $M_j = \prod_{i \neq j} V_i$. Then $G/M_j \cong V_j \rtimes \mathrm{D}_{2r}$ is irreducible and $\rho, \tau \notin M_j$.

- $\mathcal{M} = \mathsf{RotaMap}(G, \rho, \tau)$.
- $M \lhd G$ with $\rho, \tau \notin M$

Quotient rotary maps [1]

$\mathcal{M}/M := \mathsf{RotaMap}(G/M, \rho M, \tau M)$

- $\mathcal{M}_i = \mathsf{RotaMap}(G_i, \rho_i, \tau_i)$ $(i = 1, \ldots, n)$.
- $H = \langle(\rho_1, \ldots, \rho_n), (\tau_1, \ldots, \tau_n)\rangle \leqslant \prod_{i=1}^{n} G_i$.

Direct products

$\prod_{i=1}^{n} \mathcal{M}_i := \mathsf{RotaMap}\left(H, (\rho_1, \ldots, \rho_n), (\tau_1, \ldots, \tau_n)\right)$.

[0][1] J. Chen, W. Fan, C. H. Li, and Y. Z. Zhu. Coverings of groups, regular dessins, and surfaces. 2024.

Proposition

If $M_1, \ldots, M_n$ are normal subgroups of G s.t. $\bigcap_{i=1}^n M_i = 1$, then $\mathcal{M} \cong \prod_{i=1}^n (\mathcal{M}/M_i)$.

Idea of proof

Let $\varphi : G \to \prod_{i=1}^n (G/M_i)$. Then φ is injective and $\varphi(\rho) = (\rho M_1, \ldots, \rho M_n)$, $\varphi(\tau) = (\tau M_1, \ldots, \tau M_n)$. Therefore, $\varphi(G) = \langle (\rho M_1, \ldots, \rho M_n), (\tau M_1, \ldots, \tau M_n) \rangle$.

Reducible case: ψ is reducible.
$G = (V_1 \times \cdots \times V_n) \rtimes \mathrm{D}_{2r}$, $M_j = \prod_{i \neq j} V_i$. If $\mathcal{M}$ is G-rotary, then $\mathcal{M} = \prod_{i=1}^{n} \mathcal{M}/M_i$. We then reduce to the irreducible case.

Warning: What if $G/M_j \cong \mathbb{Z}_p \rtimes \mathrm{D}_{2r}$?
We have to generalize PX graphs to augmented PX graphs $\mathrm{C}^*(p, r, s, \delta)$ to include this.

- $s \geqslant 1$: $\mathrm{C}^*(p, r, s, \delta) = \mathrm{C}(p, r, s)$.
- $s = 0$: $\mathrm{C}^*(p, r, 0, -1) = \mathrm{C}_{pr}$ and $\mathrm{C}^*(p, r, 0, 1) = \mathrm{C}_r^{(p)}$.

Theorem (D., Z. Guo, L. Liu)

A PX map is a direct product of irreducible **augmented** PX maps. Conversely, a direct product of irreducible augmented PX maps is an augmented PX map.

$$\begin{array}{ccc} \mathcal{M} & = & \mathcal{M}_1 \times \cdots \times \mathcal{M}_n \\ \mathbb{Z}_p^{s+1} \rtimes_\psi \mathrm{D}_{2r} & \leqslant & (V_1 \rtimes_{\psi_1} \mathrm{D}_{2r}) \times \cdots \times (V_n \rtimes_{\psi_n} \mathrm{D}_{2r}) \\ \psi & = & \psi_1 + \cdots + \psi_n \end{array}$$

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Question: for which ψ, $G = \mathbb{Z}_p^{s+1} \rtimes_\psi \mathrm{D}_{2r}$ has a rotary map (ρ, τ) s.t. $|\rho| = 2p$?

Question: Does this decomposition of PX maps unique?

Theorem (D., Z. Guo, L. Liu)

$G = \mathbb{Z}_p^{s+1} \rtimes_\psi \mathrm{D}_{2r}$ has a rotary pair (ρ, τ) such that $|\rho| = 2p$ if and only if ψ is a multiplicity-free representations (except one case).

Warning: $\mathcal{M} = \mathcal{M} \times \mathcal{M}$.
We always assume our decomposition $\mathcal{M} = \mathcal{M}_1 \times \cdots \times \mathcal{M}_n$ satisfying $\mathcal{M}_i \ncong \mathcal{M}_j$.

Theorem (D., Z. Guo, L. Liu)

The decomposition of a PX map into irreducible augmented PX maps is unique.

This completes the proof!

- The case where $p = 2$ is partial solved [2].
- The case where $p \mid r$ is much harder. ($G = \mathbb{Z}_p^{s+1}.\mathrm{D}_{2r}$ can be non-split.)
- Can this correspondence be generalized to other graphs? (elementary abelian multicover of a graph Γ, e.g. $\Gamma = \mathrm{K}_n$.)

Thank you!

[0] [2] R. Jajcay, P. Potočnik, and S. Wilson. The praeger-xu graphs: Cycle structures, maps, and semitransitive orientations. 2019.

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